A space-time view on low-frequency electrodynamics

Stefan Kurz, Saku Suuriniemi
Tampere University of Technology
Department of Electronics
Electromagnetics
33101 Tampere, Finland
E-mail: stefan.kurz@tut.fi

Abstract

In many engineering applications, electromagnetic wave propagation can be neglected. This class of *low-frequency problems* requires that the shortest relevant wavelength is substantially greater than the diameter of the domain of the boundary value problem. In this case, the velocity of light can be regarded as infinite, which amounts to neglecting retardation: The changes in the distribution of charges and currents instantaneously propagate into the whole space. This approximation can be achieved formally with two different energy arguments:

- We can disregard the electric field energy by letting the electric permittivity tend to zero. This so-called *magneto-quasistationary* (MQS) approach, or eddy current model, describes well dominantly inductive systems. It is a good approximation in the interior and vicinity of good conductors. Faraday's law is taken into account, while displacement currents are neglected. A rigorous analysis including error estimates can be found in [1].
- In contrast, we can disregard the magnetic field energy by letting the magnetic permeability tend to zero. This so-called *electro-quasistatic* approach (EQS) describes well dominantly capacitive systems. It is a good approximation in the interior and vicinity of dielectrics, which might exhibit some losses. Displacement currents are taken into account, while Faraday's law is neglected.

More detailed considerations about the domains of validity can be found in [2].

These familiar approximations can be also approached from a quite different, structural, point of view. Four-dimensional space-time is then chosen as the backdrop. The full set of Maxwell's equations and constitutive relations is invariant with respect to the Poincaré transformation group and therefore relativistically correct in the first place. The relativistic electrodynamics has been shown [3, 4] to exhibit two Galilean limits, the two particular cases mentioned above. Therefore, the above approximations are known as low-frequency or *Galilean electrodynamics.* The virtue of such structural approach is consistency that makes it possible to rigorously study the formulations for low-frequency problems involving motion. It brings us to the question: Which are the minimal structures that have to be added to space-time in order to describe low-frequency electrodynamics consistently?

To tackle this question, we first observe that the difference between Galilean and relativistic electrodynamics is purely metrical. Pre-metric electrodynamics remains completely unaffected. We follow [5] and model space-time as a four-dimensional differentiable manifold M. The electromagnetic phenomenon is described by the 2-forms F and G, where F is the electromagnetic field and G the electromagnetic excitation. Pre-metric electrodynamics is governed by the principles of flux and charge conservation, $\mathrm{d}F = 0$, $\mathrm{d}G = J$, where J is the electric charge-current 3-form and d denotes the exterior derivative on M.

The distinctive feature of Galilean relativity is the existence of an absolute time t, which gives rise to a preferred *foliation* on M. Foliation induces *horizontal vector fields*, vector fields $\boldsymbol{v}$ that completely lie in the hypersurfaces $t =$ const of the foliation, i.e. $\mathrm{d}t|\boldsymbol{v} = 0$ holds. Finally, a *Euclidean metric* χ is introduced on the space of tangential vector fields.

To bring the four-dimensional formulation of electrodynamics into a (3+1)-dimensional decomposition, *observer structure* is introduced. An observer is modeled as an everywhere nonzero vector field $\boldsymbol{u}$ transverse to the foliation. It can always be normalized such that $\mathrm{d}t|\boldsymbol{u} = 1$ holds. The integral curves of $\boldsymbol{u}$ describe the observer's world lines. In contrast to relativistic observers, world lines do not need to intersect the foliation perpendicularly. The vector field $\boldsymbol{u}$ together with the 1-form $\mathrm{d}t$ constitutes an observer structure ($\boldsymbol{u}$, $\mathrm{d}t$), which decomposes the laws of electrodynamics

Abstract submitted May 31, 2010

in four dimensions into their (3+1)-dimensional counterparts, as desired [6]. This completes the definition of the required minimal structures.

Recall that in relativistic electrodynamics space-time is equipped with a Lorentzian metric, which gives rise to a Hodge star operator *, which in turn allows writing the constitutive relation for free space as $G = 1/Z_0{*}F$, where Z_0 is the wave impedance. In our case, the metric structure contains less information. It is possible to write a metric-induced linear operator $L(\chi)$, which maps p-forms to $(4\text{-}p)$-forms. Let μ, ε, κ denote the permeability, permittivity, conductivity metrics, respectively, and $\boldsymbol{n}$ the particular observer with respect to whom the conductor is at rest. Then, the relevant constitutive relations are

- for the MQS model $G = L(\mu)F$ and $J = L(\kappa)\mathbf{i}(\boldsymbol{n})F$,
- for the EQS model $F = L(\varepsilon)G$ and $F = L(\kappa)\mathbf{i}(\boldsymbol{n})J$,

where $\mathbf{i}(\boldsymbol{n})$ denotes the contraction with the vector field $\boldsymbol{n}$. The second equation in both cases is Ohm’s law in its appropriate space-time form. The operator $L(\chi)$ is no more an isomorphism, but has a non-trivial kernel. As a consequence, the MQS model determines the electric field in non-conducting regions up to an arbitrary gradient field only.

From the normalization condition $\mathrm{d}t|\boldsymbol{u} = 1$ it follows that two observers $\boldsymbol{u}$, $\boldsymbol{u}'$ differ at most by a horizontal vector field $\boldsymbol{v} = \boldsymbol{u}'-\boldsymbol{u}$, which locally describes their relative velocity. The transformation laws that relate the fields as perceived by the primed and unprimed observers can be easily derived from this, see e.g. [7] for the result. The transformation laws do not involve the metric, thanks to the simpler observer structure, and are therefore not restricted to uniform motion. In fact, without recourse to the metric structure we cannot even tell what is meant by uniform motion.

The transformation laws connect and encompass the Eulerian and Lagrangian descriptions of low-frequency electrodynamics of moving bodies. If the motion is restricted to rigid body motion, then the metric structure of free space remains invariant under the transformations. It is actually sufficient that the metric is preserved on each hypersurface t = const separately, so that we can allow a one-parameter family of isometries that smoothly depend on the parameter t.

We conclude that in low-frequency electrodynamics all rigid frames are equivalent. This goes beyond the standard principle of Galilean relativity, where only inertial frames can be regarded as equivalent. We emphasize that this property is restricted to low-frequency electrodynamics. This justifies the usual analysis of rotating machines from the rotor’s point of view. The same property cannot, and must not, be expected in the general case of relativistic electrodynamics, though. This is demonstrated by means of a Born rigid rotating frame within the classical paradox of Schiff [8].

References

[1] K. Schmidt, O. Sterz, and R. Hiptmair, „Estimating the Eddy-Current Modeling Error,” IEEE Trans. Magn., vol. 44(6), pp. 686-689, June 2008.

[2] T. Steinmetz, S. Kurz, and M. Clemens, „Domains of Validity of Quasistatic and Quasistationary Field Approximations,” submitted for publication in COMPEL.

[3] M. Le Bellac and J.-M. Lévy-Leblond, „Galilean Electromagnetism,” Il Nuovo Cimento B, vol. 14(2), pp. 217-234, April 1973.

[4] M. de Montigny, and G. Rousseaux, „On the electrodynamics of moving bodies at low velocities,” Eur. J. Phys., vol. 27(4), pp. 755-768, July 2006.

[5] F. Hehl and Y. Obukhov, *Foundations of Classical Electrodynamics*. Boston: Birkhäuser, 2003.

[6] S. Kurz, B. Auchmann, and B. Flemisch, „Dimensional reduction of field problems in a differential-forms framework,” COMPEL, vol. 28(4), pp. 907-921, 2009.

[7] H.H. Woodson and J.R. Melcher, *Electromechanical Dynamics*. New York: John Wiley, 1968.

[8] L. Schiff, “A question in general relativity,” Proc. Nat. Acad. Sci. USA, vol. 25, pp. 391–395, 1939.

Abstract submitted May 31, 2010