Observations about conservative quantities in electromagnetism and the computation of forces

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1 Introduction

The purpose of the discussion is to analyse how differential 3-forms (with a given physical meaning) defined on a spacetime manifold transforme under application of an arbitrary smooth map p. Although this might seem a trivial exercise at first sight, one shall see that a number of interesting conclusions can be drawn.

2 Results

Let $M = \mathbb{S}_M \times \mathbb{T}_M$ be the Cartesian product of $\mathbb{S}_M = \mathbb{R}^3$ and $\mathbb{T} = \mathbb{R}$, i.e. a manifold with a spacetime structure. Let N be a copy of M and $p : M \mapsto N$ an arbitrary differential map.

According to modern definitions, especially those rooted in Physics, differential forms can be regarded as maps defined on the sets of smooth curves, surfaces, ... defined on manifolds. Differential forms of degree 3 in the 4-dimensional manifolds M and N being forms of maximal degree minus 1, one seeks first inspiration from the case of 2-forms in a 3-dimensional manifold. The geometric notion of flux tube is recalled and the distinction between solenoidal and non solenoidal fields is explained geometrically.

The notion of flux tube is then generalized to 3-forms in 4-dimensional spacetime manifolds, and it is shown that the notion of a solenoidal field transforms into that of a conservative field. It is also shown how this extension from a space to spacetime leads to a natural notion of velocity for the quantity represented by the considered 3-form.

The reasoning is then brought one step further by considering the energy density of an electromagnetic wave (3-form). The velocity of the wave is then interpreted in the light of the previous results and shown to be independent of the mapping p.

In particular, the results described above justify using Eulerian coordinates and the Lie derivative $\mathcal{L}_{\mathbf{v}}$ to establish energy balances also in systems with electromagnetic fields. Among other benefits [1], this allows a straigthforward interpretation for the Maxwell stress tensor. If Ω represents the empty space between a number of moving objects Ω_k, $k = 1, \ldots, N$, $\partial\Omega = \sum_k \partial\Omega_k$, one has (with standard notations) :

$$\begin{aligned}\int_\Omega \mathcal{L}_{\mathbf{v}} \frac{\mathbf{b}\cdot\mathbf{h}}{2} - \int_{\partial\Omega} \mathbf{h} \times \mathcal{L}_{\mathbf{v}}\mathbf{a} &= \int_\Omega \operatorname{div}\left(\frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbf{v}\right) - \int_{\partial\Omega} \mathbf{h} \times (\operatorname{grad}(\mathbf{a}\cdot\mathbf{v}) - \mathbf{v}\times\mathbf{b}) \\ &= \int_{\partial\Omega} \{\frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbf{v} + \mathbf{b}\cdot\mathbf{h}\ \mathbf{v} - \mathbf{b}\ \mathbf{h}\cdot\mathbf{v}\} = \int_{\partial\Omega}\left(\mathbf{b}\ \mathbf{h} - \frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbb{I}\right) : \mathbf{v}\mathrm{d}\partial\Omega, \end{aligned} \tag{1}$$

i.e. the motion-induced variation of magnetic energy in Ω is equal to the motion-induced part of the Poynting Vector transferred to the N moving objects plus the mechanical work transfered to the objects by the Maxwell stress tensor.

References

[1] F. Henrotte, H. Heumann, E. Lange, K. Hameyer, *Upwind 3-D Vector Potential Formulation for Electromagnetic Braking Simulations*, IEEE Transactions on Magnetics, 46(8), August 2010.